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LETTER TO THE EDITOR

The Eckhaus equation in an external potential

F Calogero† and S De Lillo‡§

† Dipartimento di Fisica, Università di Roma 'La Sapienza', 00185 Roma, Italy, and Istituto Nazionale di Fisica Nucleare, Sezione di Roma, Roma, Italy

‡ Dipartimento di Fisica, Università di Perugia, 06100 Perugia, Italy, and Istituto Nazionale di Fisica Nucleare, Sezione di Perugia, Perugia, Italy

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Abstract. It is pointed out that the Eckhaus nonlinear PDE

$$\mathrm{i}\Psi_t+\Psi_{xx}+[2(|\Psi|^2)_x+|\Psi|^4-V(x)]\Psi=0$$

remains C-integrable (i.e., integrable by a change of dependent variable) even in the presence of any given external potential $V(x)$.

The Eckhaus nonlinear PDE

$$\mathrm{i}\Psi_t+\Psi_{xx}+[2(|\Psi|^2)x+|\Psi|^4-V(x)]\Psi=0 \qquad \Psi\equiv\Psi(x,t) \tag{1}$$

is C-integrable, namely it can be linearized, and solved, by an appropriate change of dependent variable (see below). Hitherto this property had been noted and used in the case $V(x)=0$ [1, 2]; the main purpose of this letter is to point out that (1) remains C-integrable for any given choice of the real 'external potential' $V(x)$. This observation may be of some interest in view of the 'universal' character of the Eckhaus equation (see, for instance, [3]), as well as its relevance as a most convenient 'theoretical laboratory' suitable to exhibit the rich phenomenology associated with 'solitonic' nonlinear PDEs [1, 2].

The linearizing transformation reads:

$$\varphi(x,t)=C(t)\Psi(x,t)\exp\left[\int_{a(t)}^{x}\mathrm{d}x'|\Psi(x',t)|^2\right] \tag{2a}$$

$$\Psi(x,t)=\varphi(x,t)\left[C^2(t)+2\int_{a(t)}^{x}\mathrm{d}x'|\varphi(x',t)|^2\right]^{-1/2} \tag{2b}$$

$$\dot{C}(t)/C(t)=\dot{a}(t)|\Psi(a(t),t)|^2+2\,\mathrm{Im}[\psi(a(t),t)\psi_x^*(a(t),t)] \tag{3}$$

$$\dot{C}(t)C(t)=\dot{a}(t)|\varphi(a(t),t)|^2+2\,\mathrm{Im}[\psi(a(t),t)\psi_x^*(a(t),t)] \tag{4}$$

$$\mathrm{i}\varphi_t(x,t)+\varphi_{xx}(x,t)-V(x)\varphi(x,t)=0. \tag{5}$$

§ On leave while serving as Secretary-General, Pugwash Conferences on Science and World Affairs, Geneva, London, Rome.

Here the real function $a(t)$ is arbitrary. In the special case in which $\Psi(x, t)$ and $\varphi(x, t)$ vanish (sufficiently fast) as $x \to -\infty$, to which our attention is hereafter confined, it is convenient to set $a = -\infty$, so that the transformations (2) read simply

$$\varphi(x, t) = C\Psi(x, t) \exp\left[\int_{-\infty}^{x} dx' |\Psi(x', t)|^2\right] \tag{6a}$$

$$\Psi(x, t) = \varphi(x, t)\left[|C|^2 + 2\int_{-\infty}^{x} dx' |\varphi(x', t)|^2\right]^{-1/2} \tag{6b}$$

with C a constant. Note that the linearizing transformation is independent of the external potential $V(x)$ (provided it is *real*, as we are assuming), which only enters in the (*linear*) Schrödinger equation (5) satisfied by $\varphi(x, t)$.

Let us also note that the results written above would apply without modification even if the external potential $V(x)$ were time-dependent.

For the class of solutions that vanish (sufficiently fast) at both ends, the functions $\varphi(x, t)$ and $\Psi(x, t)$ can both be normalized by setting

$$\int_{-\infty}^{+\infty} dx |\varphi(x, t)|^2 = \int_{-\infty}^{+\infty} dx |\Psi(x, t)|^2 = 1 \tag{7a}$$

provided the constant C in (6) is constrained as follows:

$$|C|^2 = 2/(e^2 - 1). \tag{7b}$$

As it is well known, for this class of solutions there holds, for the linear Schrödinger equation, the classical (Ehrenfest) theorems:

$$(d/dt) \int_{-\infty}^{+\infty} dx\, x |\varphi(x, t)|^2 = i \int_{-\infty}^{+\infty} dx \{\varphi(x, t)[\varphi_x(x, t)]^* - [\varphi(x, t)]^* \varphi_x(x, t)\} \tag{8a}$$

$$(d/dt)^2 \int_{-\infty}^{+\infty} dx\, x |\varphi(x, t)|^2 = -2 \int_{-\infty}^{+\infty} dx\, V_x(x) |\varphi(x, t)|^2. \tag{8b}$$

It is amusing to note that, for the Eckhaus equation (1), the corresponding results read as follows:

$$(d/dt) \int_{-\infty}^{+\infty} dx\, x |\Psi(x, t)|^2 = i \int_{-\infty}^{+\infty} dx \{\Psi(x, t)]^* - [\Psi(x, t)]^* \Psi_x(x, t)\} \tag{9a}$$

$$(d/dt)^2 \int_{-\infty}^{+\infty} dx\, x |\Psi(x, t)|^2 = -2 \int_{-\infty}^{+\infty} dx\, V_x(x) |\Psi(x, t)|^2 - 2 \int_{-\infty}^{+\infty} dx [(|\Psi(x, t)|)_x]^2. \tag{9b}$$

In the special case corresponding to a harmonic oscillator potential

$$V(x) = x^2 \tag{10}$$

the general solution of the Schrödinger equation (5) reads of course

$$\varphi(x, t) = \sum_{n=0}^{\infty} c_n H_n(x) \exp[-\tfrac{1}{2}x^2 - i(2n+1)t] \tag{11}$$

where the Hermite polynomial $H_n(x)$ satisfies the ODE

$$H_n''(x) - 2xH_n' + 2nH_n(x) = 0. \tag{12}$$

The corresponding general solution of the Eckhaus equation (1) is, of course, given by (2*b*) with (11). In the special case in which $c_n = \delta_{nm}$, so that

$$|\varphi(x,t)|^2 = c_m^2 H_n^2(x) \exp(-x^2) \tag{13}$$

is time-independent, the corresponding (static) solution $\Psi(x,t)$ has the same property, namely

$$|\Psi(x,t)|^2 = c_m^2 H_m^2(x) \exp(-x^2) \left[C^2 + c_m^2 \int_{-\infty}^{x} \mathrm{d}y\, H_m^2(y) \exp(-y^2) \right]^{-1} \tag{14}$$

is also time-independent (this is clearly a general property). However, while in this case $|\varphi(x,t)|^2$ is symmetric in x, $|\varphi(x,t)| = |\varphi(-x,t)|$ (see (13)), $|\Psi(x,t)|$ does not possess this property (see (14)).

Let us also note that, while in the case of the harmonic oscillator potential (10) the mean value $\bar{x}(t)$ of the coordinate x with respect to the 'wavefunction' $\varphi(x,t)$, satisfying the *linear* Schrödinger equation (5) (with (10)),

$$\bar{x}(t) = \int_{-\infty}^{+\infty} \mathrm{d}x\, x |\varphi(x,t)|^2 \tag{15}$$

oscillates harmonically:

$$\bar{x}(t) = \bar{x}(0)\cos(2t) + \frac{1}{2}\dot{\bar{x}}(0)\sin(2t) \tag{16}$$

(as a well known general consequence of (8*b*)), no analogous result holds for the mean value $\tilde{x}(t)$ of x taken with respect to the 'wavefunction' $\Psi(x,t)$ satisfying the Eckhaus equation (1) (with (10)),

$$\tilde{x}(t) = \int_{-\infty}^{+\infty} \mathrm{d}x\, x |\Psi(x,t)|^2. \tag{17}$$

As a second example, let us mention the case

$$V(x) = -2p\delta(x-a) \tag{18}$$

with p a *positive* constant. It is then well known that the linear Schrödinger equation (5) has one static normalized solution ('bound state'),

$$\varphi(x,t) = p^{1/2} \exp[p(x-a) + \mathrm{i}p^2 t] \qquad x \leqslant a \tag{19a}$$

$$\varphi(x,t) = p^{1/2} \exp[p(a-x) + \mathrm{i}p^2 t] \qquad x \geqslant a. \tag{19b}$$

In this case the corresponding normalized static solution for Ψ can be explicitly evaluated:

$$\Psi(x,t) = p^{1/2} \exp[p(x-a) + \mathrm{i}p^2 t]/\{C^2 + \exp[2p(x-a)]\}^{1/2} \qquad x \leqslant a \tag{20a}$$

$$\Psi(x,t) = p^{1/2} \exp[p(x-a) + \mathrm{i}p^2 t]/\{C^2 + 2 - \exp[2p(a-x)]\}^{1/2} \qquad x \geqslant a \tag{20b}$$

with the constant C satisfying (7*b*). It is left as an amusing and instructing exercise for the diligent reader to generalize the results of [1] in this case.

Finally we note that the extension of the Eckhaus equation described in this letter is applicable, in obviously analogous manner, to other C-integrable equations, such as those discussed in [3] (see also [4, 5]).

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